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ON A COROLLARY OF THE NOETHER THEOREM FOR THE TWO-DIMENSIONAL PROBLEM OF THE MAYER TYPE *

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The group-theoretical approach to the construction of differential laws of conservation is considered for two-dimensional Mayer type problems. Approach is based on the use of the Lie—Ovsiannikov theory and the Noether theorem /4,5/ which is the fundamental tool for deriving the laws of conservation for uncontrolled physical processes (see, e.g., /6—9/). Certain classes of optimally controlled processes were earlier considered in /10—12/ from the point of view of the Noether theory. Sufficient conditions of existence of first integrals of two-dimensional variational problems of the Mayer type are obtained. As an example, the problem of heat transfer optimization in the boundary layer of a compressible gas is considered.

1. The Noether theorem for the Mayer invariant problem. Let us consider the functional

$$J = \oint_l f(s, u^k, u_s^k)\, ds \quad (k = 1, \ldots, n) \tag{1.1}$$

where s is the length of arc; $u_s^k = du^k / ds$; $u^k(x, y)$ are continuously differentiable functions which satisfy in some simply connected region D the differential relations

$$\Psi^j \equiv (a_{ik}^j u^k + a_i^j)\, u_x^i + (b_{ik}^j u^k + b_i^j)\, u_y^i + c_k^j u^k + c^j = 0, \qquad (i, j, k = 1, \ldots, n) \tag{1.2}$$

where $a_{ik}^j, a_i^j, b_{ik}^j, b_i^j, c_k^j$, and c^j are continuously differentiable functions of variables x and y. Along the contour l are specified the isoperimetric

$$\Gamma^i = \oint_l g^i(s, u^k, u_s^k)\, ds \quad (i = 1, \ldots, m) \tag{1.3}$$

and boundary conditions

$$\varphi^j(s, u^k, u_s^k) = 0 \quad (j = 1, \ldots, p < n + 1) \tag{1.4}$$

Note that the number of degrees of freedom of the variational problem is in region D equal zero. The boundary l of region D and the boundary values of the sought functions $u^k(x, y)$, where point $(x, y) \in l$ /13—15/, are taken as the control.

Use of the Lagrange formalism leads to the investigation for extremum of the functional

$$V = V_1 + V_2, \quad V_1 = \oint_l F dx, \quad V_2 = \iint_D L\, dx\, dy \tag{1.5}$$

$$F = f + p_i g^i + q_j \varphi^j, \quad L = \lambda_k \Psi^k \quad (i = 1, \ldots, m; \quad j = 1, \ldots, p; \; k = 1, \ldots, n)$$

where p_i, q_j, and λ_k are Lagrange multipliers.

It was shown in /16/ that the necessary and sufficient condition of invariance of the integral V_2 with respect to group G_r generated by the Lie algebra of infinitesimal operators

$$X = e^\alpha X_\alpha = \xi_x \frac{\partial}{\partial x} + \xi_y \frac{\partial}{\partial y} + \xi_{u^k} \frac{\partial}{\partial u^k} + \xi_{\lambda^k} \frac{\partial}{\partial \lambda^k}, \qquad (\alpha = 1, \ldots, r; \; k = 1, \ldots, n) \tag{1.6}$$

where $e^1, \ldots, e^r$ are coordinates of the direction vector of one-parameter group, and X_α are basis vectors, is of the form

$$X^* L = 0, \quad X^* = X + \xi_{u_x}^k \frac{\partial}{\partial u_x^k} + \xi_{u_y}^k \frac{\partial}{\partial u_y^k} \tag{1.7}$$

where X^* is the operator of the continued group whose coordinates are calculated by formulas of the theory of continuation /3/

$$\xi_{u_x^k} = D_x \xi_u^k - u_x^k D_x \xi_x - u_y^k D_x \xi_y, \qquad \xi_{u_y^k} = D_y \xi_u^k - u_x^k D_y \xi_x - u_y^k D_y \xi_y \tag{1.8}$$

$$D_x = \frac{\partial}{\partial x} + u_x^k \frac{\partial}{\partial u^k} + \lambda_x^k \frac{\partial}{\partial \lambda^k}, \quad D_y = \frac{\partial}{\partial y} + u_y^k \frac{\partial}{\partial u^k} + \lambda_y^k \frac{\partial}{\partial \lambda^k}$$

It follows from /11/ that the functional V_2 is invariant with respect to group G_r° admitted by system (1.2). We represent the determining equations for the coordinates of that

*Prikl.Matem.Mekhan.,44,No.3,448-453,1980 group in the form /3/

$$\{\xi_x\Lambda_x^{\ j} + \xi_y\Lambda_y^{\ j} + \xi_u^{\ k}P_u^{\ j} + \xi_{u_x}^{\ ok}(a_{ks}^{\ j}u^s + a_k^{\ j}) + \xi_{u_y}^{\ ok}(b_{ks}^{\ j}u^s + b_k^{\ j})\}_{(1.2)} = 0 \tag{1.9}$$

$$\Lambda_x^{\ j} = \left(\frac{\partial a_{ik}^j}{\partial x}u^k + \frac{\partial a_i^{\ j}}{\partial x}\right)u_x^{\ i} + \left(\frac{\partial b_{ik}^j}{\partial x}u^k + \frac{\partial b_i^{\ j}}{\partial x}\right)u_y^{\ i} + \frac{\partial c_k^{\ j}}{\partial x}u^k + \frac{\partial c^j}{\partial x}$$

$$\Lambda_y^{\ j} = \left(\frac{\partial a_{ik}^j}{\partial y}u^k + \frac{\partial a_i^{\ j}}{\partial y}\right)u_x^{\ i} + \left(\frac{\partial b_{ik}^j}{\partial y}u^k + \frac{\partial b_i^{\ j}}{\partial y}\right)u_y^{\ i} + \frac{\partial c_k^{\ j}}{\partial y}u^k + \frac{\partial c^j}{\partial y}$$

$$P_u^{\ j} = a_{ik}^{\ j}u_x^{\ i} + b_{ik}^{\ j}u_y^{\ i} + c_k^{\ j};\ i,\ j,\ k,\ s = 1,\ \ldots,\ n$$

where $\xi_{u_x}^{\ ok}$ and $\xi_{u_y}^{\ ok}$ are determined by formulas (1.8) in which the total differentiation operators

$$D_x^{\circ} = \frac{\partial}{\partial x} + u_x^{\ k}\frac{\partial}{\partial u^k},\quad D_y^{\circ} = \frac{\partial}{\partial y} + u_y^{\ k}\frac{\partial}{\partial u^k} \tag{1.10}$$

are substituted for operators D_x and D_y.

According to /5,12/ the following theorem holds for the quasilinear relationships (1.2).

The Noether theorem. The invariance of functional V_2 with respect to group G_r° implies the law of conservation

$$D_x\{\lambda_j(a_{ks}^{\ j}u^s + a_k^{\ j})(\xi_u^{\ k} - u_x^{\ k}\xi_x - u_y^{\ k}\xi_y)\} + D_y\{\lambda_j(b_{ks}^{\ j}u^s + b_k^{\ j})(\xi_u^{\ k} - u_x^{\ k}\xi_x - u_y^{\ k}\xi_y)\} = 0 \tag{1.11}$$

We say that the variational problem (1.1)—(1.4) admits the first integral, if Eq. (1.11) is integrable exactly once with respect to the x- or y-coordinate.

2. Sufficient conditions of existence of first integrals. The construction of laws of conservation (1.11) necessitates the determination of the coordinate of the infinitesimal operator of group G_r° by solving the determining equations (1.9). Local Lie groups which are admitted by system (1.2) in particular cases are known. They can be used in the formulation of respective variational problems of derivation of conservation laws. Below, we shall consider the case frequently encountered in applications, when all coefficients of system (1.2) are independent of x or y. We assume for definiteness that

$$\frac{\partial a_{ik}^j}{\partial y} = \frac{\partial a_i^{\ j}}{\partial y} = \frac{\partial b_{ik}^j}{\partial y} = \frac{\partial b_i^{\ j}}{\partial y} = \frac{\partial c_k^{\ j}}{\partial y} = \frac{\partial c^j}{\partial y} \equiv 0,\quad (i, j, k = 1, \ldots, n) \tag{2.1}$$

It now follows from Eqs. (1.9) that $\xi_x = \xi_u^{\ k} \equiv 0$, and $\xi_y = \varepsilon$, i.e. that the functional V_2 is invariant with respect to the group of transfers on coordinate y. If some functional in a variational problem with a single independent variable is invariant with respect to a group of transfers relative to that variable, it is possible to obtain the first integral of the variational problem /4/. But in the two-dimensional case the invariance of functional V_2 with respect to a group of transfers on coordinate x or y does not necessarily guarantee the existence of the first integral.

If relations (1.2) do not contain terms with derivatives of u_x^{α} of some fixed α (such as, for example, boundary layer equations /17/), then by virtue of (2.1) the conservation law (1.11) assumes the form

$$D_xA + D_yB = 0 \tag{2.2}$$

$$A = u_y^{\ k}\lambda_j(a_{ks}^{\ j}u^s + a_k^{\ j})(1 - \delta_\alpha^{\ k}) \tag{2.3}$$

$$B = u_y^{\ k}\lambda_j(b_{ks}^{\ j}u^s + b_k^{\ j})\ (k,\ j,\ s = 1,\ \ldots,\ n) \tag{2.4}$$

where $\delta_\alpha^{\ k}$ is the Kronecker delta.

The Euler—Lagrange—Ostrogradskii equation for function u^α (obtained by equating to zero the expression at variation δu^α) becomes

$$\lambda_j(a_{i\alpha}^{\ j}u_x^{\ i}(1 - \delta_\alpha^{\ i}) + b_{i\alpha}^{\ j}u_y^{\ i} + c_\alpha^{\ j}) - D_y\{\lambda_j(b_{\alpha i}^{\ j}u^i + b_\alpha^{\ j})\} = 0 \tag{2.5}$$

We stipulate that

$$a_{i\alpha}^{\ j} = b_{i\alpha}^{\ j} = c_\alpha^{\ j} \equiv 0\quad (i,\ j = 1,\ \ldots,\ n) \tag{2.6}$$

Then formulas (2.5) and (2.3) assume respectively form

$$\lambda_j b_{i\alpha}^{\ j}u_y^{\ i} - b_\alpha^{\ j}\partial\lambda_j/\partial y = 0 \tag{2.7}$$

$$A = \lambda_j(1 - \delta_\alpha^{\ s})[u^s a_{is}^{\ j}u_y^{\ i}(1 - \delta_\alpha^{\ i}) + u_y^{\ s}a_s^{\ j}] \tag{2.8}$$

where $(i,\ j,\ s = 1,\ \ldots,\ n)$.

Let for some fixed $p \neq \alpha$

$$a_{is}^{\ j} \equiv 0,\ (a_{ip}^{\ j})^2\delta_i^{\ i}\delta_j^{\ j} \neq 0,\ b_{i\alpha}^{\ j} = a_{ip}^{\ j} \tag{2.9}$$

$$a_s^j \equiv 0,\ (a_p^j)^2\delta_j^j \neq 0,\ b_\alpha^j = a_p^j \tag{2.10}$$

Then from (2.8) by virtue of (2.7) we have

$$A = D_y\{b_\alpha^j u^p \lambda_j\}$$

From (2.2) we now have

$$D_x\{b_\alpha^j u^p \lambda_j\} + u_y^k \lambda_j b_{ki}^j u^i (1 - \delta_\alpha^k) + u_y^k \lambda_j b_k^j = g(x) \tag{2.11}$$

where $g(x)$ is an arbitrary integration function which is determined by the transversality conditions /14,15/ with allowance for the boundary conditions (1.4).

Thus we have the following corollary of the Noether theorem.

Corollary. If system (1.2) admits the one-parameter group $x' = x, y' = y + \varepsilon$, and $u^{k'} = u^k$ conditions $a_{\alpha k}^j = a_\alpha^j = 0$ and, also, conditions (2.6), (2.9), and (2.10) are satisfied, the variational problem (1.1)—(1.4) admits integral (2.11).

3. Optimization of heat transfer in the boundary layer of compressible gas.

Equations of compressible gas boundary layer whose coefficient of viscosity linearly depends on temperature expressed in the Dorodnitsyn—Stewartson variables ξ and η is of the form /17/

$$U\frac{\partial U}{\partial \xi} + V\frac{\partial U}{\partial \eta} - U_e\frac{\partial U_e}{d\xi}(1+S) = \nu_1\frac{\partial^2 U}{\partial \eta^2}, \qquad \frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta} = 0 \tag{3.1}$$

$$U\frac{\partial S}{\partial \xi} + V\frac{\partial S}{\partial \eta} = \nu_1\left\{\frac{1}{\sigma}\frac{\partial^2 S}{\partial \eta^2} - \frac{1-\sigma}{\sigma}\frac{(k-1)M_e^2/2}{1+(k-1)M_e^2/2}\frac{\partial^2}{\partial \eta^2}\left(\frac{U}{U_e}\right)^2\right\}$$

We assume the boundary conditions in the form

$$\begin{aligned} &\eta = 0, \quad U = 0, \quad V = V_0(\xi), \quad S = S_0, \\ &\eta \to \infty, \quad U \to U_e(\xi), \quad S \to 0, \\ &\xi = 0, \quad U = U(\eta), \quad S = S(\eta) \end{aligned} \tag{3.2}$$

where $U(\xi, \eta)$ and $V(\xi, \eta)$ are the longitudinal and transverse velocity components in an imaginary stream, $S(\xi, \eta)$ is the enthalpy, ν is the kinematic viscosity coefficient, σ is the Prandtl number, and k is the adiabatic constant. The subscript 1 relates in (3.1) and (3.2) denotes an adiabatically and isentropically frozen state of gas in the external stream, and subscript e relates to the external stream.

We have the following optimal problem /15/. Find among the continuous controls $V_0(\xi)$ that satisfy Eqs. (3.1) and boundary conditions (3.2) one that ensures the minimum quantity of heat

$$Q = -\int_0^L k\left(\frac{\partial S}{\partial \eta}\right)_{\eta=0} d\xi \tag{3.3}$$

transmitted from the hot gas to the wall surface with a given capacity of the cooling system

$$N = \int_0^L aV_0^2 d\xi \tag{3.4}$$

where a is a known constant.

Using the notation

$$U = U^1,\ V = U^2,\ S = U^3,\quad \frac{\partial U^1}{\partial \eta} = U^4,\ \frac{\partial U^3}{\partial \eta} = U^5,\ \frac{\partial}{\partial \eta}\left(\frac{U}{U_e}\right)^2 = U^6$$

we represent system (3.1) in the equivalent form

$$\Psi^1 \equiv U^1\frac{\partial U^1}{\partial \xi} + U^2\frac{\partial U^1}{\partial \eta} - U_e\frac{dU_e}{d\xi}(1+U^3) - \nu_1\frac{\partial U^4}{\partial \eta} = 0, \qquad \Psi^2 \equiv \frac{\partial U^1}{\partial \xi} + \frac{\partial U^2}{\partial \eta} = 0, \quad \Psi^3 \equiv U^4 - \frac{\partial U^1}{\partial \eta} = 0 \tag{3.5}$$

$$\Psi^4 \equiv U^1\frac{\partial U^3}{\partial \xi} + U^2\frac{\partial U^3}{\partial \eta} - \nu_1\left\{\frac{1}{\sigma}\frac{\partial U^5}{\partial \eta} - \frac{1-\sigma}{\sigma} \times \frac{(k-1)M_e^2/2}{1+(k-1)M_e^2/2}\frac{\partial U^6}{\partial \eta}\right\} = 0$$

$$\Psi^5 \equiv U^5 - \frac{\partial U^3}{\partial \eta} = 0, \quad \Psi^6 \equiv U^6 - 2\frac{U^1}{U_e^2}\frac{\partial U'}{\partial \eta} = 0$$

The basic functional is of the form

$$V_2 = \iint_D \lambda_j \Psi^j d\tau \quad (j = 1, \ldots, 6)$$

where D is the region bounded by lines

$$\eta = 0, \quad \xi = 0, \quad \xi = L, \quad \eta \to \infty$$

Theorem. The variational problem (3.1)—(3.4) of boundary layer optimal control for any $U_e(\xi)$ admits the first integral.

Proof. In the notation of Sect.1 we have

$$a_{11}^1 = 1,\quad b_{12}^1 = 1,\quad b_4^1 = -\nu_1,\quad c_3^1 = -U_e\frac{dU_e}{d\xi},\quad c^1 = -U_e\frac{dU_e}{d\xi},\quad a_1^2 = 1,\quad b_2^2 = 1,\quad b_1^3 = -1,\quad c_4^3 = 1 \tag{3.6}$$

$$a_{31}^4 = 1,\quad b_{32}^4 = 1,\quad b_5^4 = -\frac{\nu_1}{\sigma},\quad b_6^4 = \nu_1\frac{1-\sigma}{\sigma}\frac{(k-1)M_e^2/2}{1+(k-1)M_e^2/2},\quad b_3^5 = -1,\quad c_5^5 = 1,\quad b_{11}^6 = -\frac{2}{U_e^2},\quad c_6^6 = 1$$

The remaining coefficients are zero. It follows from (3.6) that $\alpha = 1$ and $p = 2$, hence all conditions of the corollary are satisfied. Consequently in conformity with (2.11) there exists the first integral

$$\frac{\partial}{\partial\xi}(U^1\lambda_2) + T_i\frac{\partial U^i}{\partial\eta} = g(\xi)\quad (i = 1,\ldots,6),\qquad T_1 = \lambda_1 U^2 - \lambda_3 - 2U^1\lambda_6 / U_e^2 \tag{3.7}$$

$$T_2 = \lambda_2,\quad T_3 = \lambda_4 U^2 - \lambda_5,\quad T_4 = -\nu_1\lambda_1,\qquad T_5 = -\sigma^{-1}\nu_1\lambda_4,\quad T_6 = \sigma^{-1}(1-\sigma)\nu_1\lambda_4\frac{(k-1)M_e^2/2}{1+(k-1)M_e^2/2}$$

Euler—Lagrange—Ostrogradskii equations and the boundary conditions are of the form

$$-U^1\frac{\partial\lambda_1}{\partial\xi} - \frac{\partial}{\partial\eta}(\lambda_1 U^2) - \frac{\partial\lambda_2}{\partial\xi} + \lambda_4\frac{\partial U^3}{\partial\xi} + \frac{\partial\lambda_3}{\partial\eta} + \frac{2U^1}{U_e^2}\frac{\partial\lambda_6}{\partial\eta} = 0 \tag{3.8}$$

$$\lambda_1\frac{\partial U^1}{\partial\xi} + \lambda_4\frac{\partial U^3}{\partial\eta} - \frac{\partial\lambda_2}{\partial\eta} = 0,\qquad \lambda_1 U_e\frac{dU_e}{d\xi} + \frac{\partial}{\partial\xi}(\lambda_4 U^1) + \frac{\partial}{\partial\eta}(\lambda_4 U^2 - \lambda_5) = 0$$

$$\lambda_3 + \nu_1\frac{\partial\lambda_1}{\partial\eta} = 0,\quad \lambda_5 + \frac{\nu_1}{\sigma}\frac{\partial\lambda_4}{\partial\eta} = 0,\quad \lambda_6 - \nu_1\frac{1-\sigma}{\sigma}\times\frac{(k-1)M_e^2/2}{1+(k-1)M_e^2/2}\frac{\partial\lambda_4}{\partial\eta} = 0$$

$$\eta = 0,\ 2\lambda_0 a V_0^2 - \lambda_2 = 0,\ \lambda_4 = -k\sigma/\nu_1,\ \lambda_1 = 0,\qquad \eta\to\infty,\ \lambda_1\to 0,\ \lambda_2\to 0,\ \lambda_4\to 0,\qquad \xi = L,\ \lambda_1 = \lambda_2 = \lambda_4 = 0 \tag{3.9}$$

Assuming the existence of $\lim_{\eta\to\infty}\partial\lambda_1/\partial\eta$ and $\lim_{\eta\to\infty}\partial\lambda_4/\partial\eta$, by virtue of which from the boundary conditions in (3.9) and the last three equations (3.8), we obtain from Eq. (3.7) $g(\xi)\equiv 0$.

We note in conclusion that a direct application of integral (3.7) for determining the optimal control $V_0(\xi)$ is the subject of special investigation. Here, we shall only point out that the first integrals of variational problems of optimal control of incompressible fluid laminar boundary layer obtained earlier /10,18/ had substantially eased the task of finding respective optimal controls.

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